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EXTENSIONS TO THE SOLUTION OF THE GRAETZ PROBLEM

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Abstract—In the original work of Graetz on heat transfer in laminar flow, the simplifying assumption was made that axial diffusion could be ignored in comparison with the other terms in the equation. This gives results which are sufficiently accurate for the large values of the Péclét number occurring with classical fluids, but the assumption cannot be applied to heat transfer in liquid metals where axial diffusion plays a significant role. Previous papers taking diffusion into account have involved separate computations for each value of the Péclét number, whereas in this paper the eigenvalues are given in the form of an asymptotic expansion so that the required values can be calculated in a simple fashion. The solution given here also takes into account preheating of the incoming fluid, and shows that this has a significant effect.

INTRODUCTION

THE RATE of heat transfer to fluids flowing in pipes has been the subject of investigation for over eighty years. Graetz [1, 2] and Nusselt [3] derived the first solutions for flow in a circular tube by ignoring the effects of axial diffusion on the temperature distribution, and calculated the first few values of the eigenvalues and the corresponding coefficients of the expansion. Subsequent papers have been concerned mainly with improved techniques for evaluating these eigenvalues (e.g. Abramowitz [4]) or applying Graetz method to other geometries. Singh [5] considered the effects of axial conduction, but his method does not allow a very great accuracy to be obtained. Furthermore, in spite of the inclusion of axial conduction, his incoming fluid is assumed to have a uniform temperature at the origin, so that the coefficients that he obtains tend to be too high.

In the present paper, the problem of heat transfer for fully developed Poiseuille flow in a circular tube has been studied under the assumption that the walls of the tube are kept at one constant temperature upstream of the origin, and at a different constant temperature downstream. In this way account has been taken of the preheating of the incoming fluid. A theoretical solution for the temperature distribution both upstream and downstream is given, but numerical solutions have only been obtained for the downstream portion because of insurmountable analytic problems associated with the upstream solution. These calculations are compared with previous results.

ANALYSIS

The physical situation considered is an infinite tube of circular cross-section through which is flowing a fluid in steady Poiseuille flow. One semi-infinite half of the tube is maintained at a temperature T_0 and the other half at temperature T_1. As usual, the physical parameters, diffusivity, density, etc., are assumed to be independent of the temperature.

If the tube has radius R, the governing diffusion-convection equation is

$$K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2}\right) = 2u_m(1 - r^2/R^2)\frac{\partial T}{\partial x}$$

with boundary conditions

$$T(R, x) = T_0, \qquad x < 0;$$
$$T(R, x) = T_1, \qquad x > 0;$$
$$T(r, x) \to T_0, \qquad x \to -\infty;$$
$$T(r, x) \to T_1, \qquad x \to \infty.$$

Here K is the thermal diffusivity, u_m is the mean fluid velocity, r is a radial variable and x is an axial variable, Since the problem is axisymetric, there is no angular variable present.

The equation is made non-dimensional by introducing the new variables

$$\theta = (T - T_0)/(T_1 - T_0),$$
$$\rho = r/R,$$
$$\xi = x/(R.Pe),$$

where Pe is the non-dimensional Péclét number $u_m . R/K$. The diffusion convection equation now becomes

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial\xi^2} = 2(1 - \rho^2)\frac{\partial\theta}{\partial\xi} \qquad (1.1)$$

$$\theta(1, \xi) = 0, \qquad \xi < 0;$$
$$\theta(1, \xi) = 1, \qquad \xi > 0;$$
$$\theta(\rho, \xi) \to 0, \qquad \xi \to -\infty;$$
$$\theta(\rho, \xi) \to 1, \qquad \xi \to +\infty.$$

Equation (1.1) can be solved formally by using a two-sided Laplace Transform in ξ. Writing

$$\bar{\theta}(\rho, \eta) = \int_{-\infty}^{\infty} \theta(\rho, \xi)\, e^{-\xi\eta}\, d\xi,$$

we obtain

$$\frac{d^2\bar{\theta}}{d\rho^2} + \frac{1}{\rho}\frac{d\bar{\theta}}{d\rho} - 2\eta(1 - \rho^2)\bar{\theta} = (-\eta^2/Pe^2)\bar{\theta} \quad (1.2)$$

$$\theta(1, \eta) = 1/\eta.$$

This equation has a regular singular point at $\rho = 0$ with indices 0, 0. We can therefore specify uniquely a solution $f(\rho, \eta)$ of the differential equation which satisfies $f(0, \eta) = 1$. Since the second solution has a logarithmic singularity in ρ at $\rho = 0$, it can be discarded. The required solution for $\bar{\theta}$ is then

$$\bar{\theta}(\rho, \eta) = \frac{1}{\eta}\frac{f(\rho, \eta)}{f(1, \eta)}.$$

Using the inverse Laplace transform we then obtain

$$\theta(\rho, \xi) = 1 + \sum_{n=0}^{\infty} \frac{e^{p_n\xi}}{p_n} \frac{f(\rho, p_n)}{\frac{\partial f}{\partial\eta}(1, p_n)}, \qquad \xi > 0;$$
$$= -\sum_{n=0}^{\infty} \frac{e^{q_n\xi}}{q_n} \frac{f(\rho, q_n)}{\frac{\partial f}{\partial\eta}(1, q_n)}, \qquad \xi < 0; \qquad (1.3)$$

where $0 > p_0 > p_1 \ldots$ are the negative zeros of $f(1, \eta)$, and $0 < q_0 < q_1 \ldots$ are the positive zeros. The negative zeros, with which this paper is chiefly concerned, are the generalization of the eigenvalues which appear in Graetz's solution, and, in the notation usually adopted in that case, we have

$$p_n \sim \tfrac{1}{2}\beta_n^2 \quad \text{as} \quad Pe \to \infty.$$

The determination of the asymptotic expansions for the p_n depends on finding the equivalent asymptotic expansion of the function $f(\rho, \eta)$.

THE ASYMPTOTIC EXPANSIONS

If we write

$$f(\rho, \eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(Pe)^{2n}} f_n(\rho, \eta)$$

then the functions f_n satisfy the recurrence equations

$$\frac{d^2 f_0}{d\rho^2} + \frac{1}{\rho}\frac{df_0}{d\rho} - 2\eta(1 - \rho^2) f_0 = 0 \qquad (2.1)$$

$$\frac{d^2 f_n}{d\rho^2} + \frac{1}{\rho}\frac{df_n}{d\rho} - 2\eta(1 - \rho^2) f_n = \eta^2 f_{n-1}. \qquad (2.2)$$

A classic Frobenius approach to equation (2.1) indicates that we can write

$$f_0(\rho, \eta) = \sum_{r=0}^{\infty} \eta^r \sum_{m=0}^{r} (-1)^m B_{m,r}\, \rho^{2m+2r}.$$

Substituting this expansion in equation (2.1) we obtain

$$2(m + r)^2 B_{m,r} = B_{m,r-1} + B_{m-1,r-1},$$

where

$$B_{0,0} = 1, \qquad B_{m,0} = 0, \qquad m \neq 0.$$

A formal Frobenius expansion of equation (1.2) suggests, and substitution in (2.2) confirms, that the other functions are given by

$$f_n(\rho, \eta) = \eta^{2n} \sum_{r=0}^{\infty} \eta^r \sum_{m=0}^{r} (-1)^m \rho^{2n+2m+2r} \times \frac{(r-m+1)\ldots(r-m+n)}{2^n n!} B_{m,r+n}.$$

Table 1. $B_{m,r}$. The notation E-1 (etc) indicates that the numbers given should be multiplied by 10^{-1} (etc)

$r =$ 0	1	2 E-1	3 E-2	4 E-3	5 E-5	6 E-7	7 E-9	8 E-11	9 E-13	10 E-15
m										
0 1·0	0·500	0·625000	0·347222	0·108507	0·217014	0·301408	0·307559	0·240281	0·148322	0·074161
1	0·125	0·347222	0·303819	0·130208	0·331549	0·559758	0·672786	0·605152	0·422716	0·235966
2		0·039063	0·077257	0·052927	0·186873	0·405017	0·595540	0·634163	0·512114	0·324594
3			0·005425	0·008437	0·047941	0·144947	0·274982	0·359720	0·345098	0·253613
4				0·000424	0·005470	0·026705	0·070931	0·120109	0·141961	0·124250
5					0·000212	0·002348	0·010088	0·023970	0·036755	0·039715
6						0·000074	0·000716	0·002756	0·005939	0·008339
7							0·000019	0·000163	0·000570	0·001126
8								0·000004	0·000029	0·000092
9									0·000001	0·000004

Table 2. Coefficients of F_0, F_1, F_2

r	$c_{0,r}$	$c_{1,r}$	$c_{2,r}$
0	1·000000 E + 0	0·250000 E + 0	0·156250 E − 1
1	0·375000 E + 0	0·451389 E − 1	0·184462 E − 1
2	0·316840 E − 1	0·255642 E − 2	0·783360 E − 4
3	0·115234 E − 2	0·704102 E − 4	0·173381 E − 5
4	0·232128 E − 4	0·114540 E − 5	0·236378 E − 7
5	0·296549 E − 6	0·122975 E − 7	0·218792 E − 9
6	0·261516 E − 8	0·936524 E − 10	0·146606 E − 11
7	0·168719 E − 10	0·532190 E − 12	0·744399 E − 14
8	0·830754 E − 13	0·234306 E − 14	0·296388 E − 16
9	0·322414 E − 15	0·822741 E − 17	0·950364 E − 19
10	0·101157 E − 17	0·235790 E − 19	0·250704 E − 21

The values of $B_{m,r}$, $r = 0, 10$, are listed in Table 1. Putting $\rho = 1$, we now obtain expansions of the form

$$f_n(1, \eta) = \eta^{2n} \sum_{r=0}^{\infty} C_{n,r} \eta^r.$$

The coefficients $C_{n,r}$ for $n = 0, 1, 2$ are listed in Table 2. For convenience we write

$$f_n(1, \eta) = F_n(\eta).$$

If we substitute the expansion

$$p_n = p_{n0} + \frac{1}{Pe^2} p_{n1} + \frac{1}{Pe^4} p_{n2} + \ldots$$

in the expression

$$F(\eta) = \sum_{r=0}^{\infty} \frac{(-1)^r}{Pe^{2r}} F_r(\eta),$$

and equate to zero, we obtain

$$F_0(p_{n0}) = 0,$$

(so that $p_{n0} = -\frac{1}{2}\beta_n^2$ as previously stated)

$$p_{n1} = F_1(p_{n0})/F_0'(p_{n0}),$$

and

$$p_{n2} = (p_{n1}F_1'(p_{n0}) - \tfrac{1}{2}(p_{n1})^2 F_0''(p_{n0}) - F_2(p_{n0}))/F_0'(p_{n0}).$$

These values, for $n = 0, 7$, are listed in Table 3.

RESULTS AND CONCLUSIONS

These values were used to obtain the results listed in Tables 4 and 5. As is usual, the mean mixed temperature θ_M is defined as

Table 3.

n	p_{n0}	p_{n1}	p_{n2}
0	−3·65679	8·36578	−38·0824
1	−22·3048	362·224	−11672·4
2	−56·9605	2493·62	−217319
3	−107·620	9176·41	−1563600
4	−174·282	24558·8	−6941700
5	−256·944	59180·2	−22997800
6	−355·608	104983	−62589100
7	−470·272	185697	−152933000

Table 4. *Eigenvalues*

$P_e =$	100	500	1000
p_0	−3·65595	−3·65676	−3·65678
p_1	−22·2687	−22·3034	−22·3044
p_2	−56·7133	−56·9505	−56·9580
p_3	−106·718	−107·583	−107·611
p_4	−171·896	−174·184	−174·257
p_5	−251·756	−256·728	−256·890
p_6	−345·736	−355·189	−355·503
p_7	−453·232	−469·532	−470·086

$$\theta_M = 4 \int_0^1 (\rho - \rho^3)\, \theta(\rho, \xi)\, d\rho,$$

and the Nusselt number Nu as

$$Nu = 2 \left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=1} / (1 - \theta_M).$$

Calculation shows that for $\xi > 0{\cdot}3$ and $Pe > 100$, the Nusselt number is given by the formula

$$Nu = 3{\cdot}657(1 + 0{\cdot}6072\, e^{(p_1 - p_0)\xi}),$$

where dependence on Pe is limited to the values of p_1 and p_0.

Also included in Table 5 are the results given by Singh [5]. Singh assumed that the fluid had uniform temperature at $\xi = 0$, an assumption which becomes less valid as the Péclét number decreases. This is reflected in the comparative discrepancies between the results. There is less variation in Nu between $Pe = 100$ and $Pe = 1000$ in Singh's results than appears if we assume preheating, and Singh's results are also consistently higher. The variation is also more pronounced near $\xi = 0$ than downstream, as would be expected.

In conclusion, it appears that the effects of preheating of the incoming fluid should not be neglected in making calculations with axial conduction included.

REFERENCES

1. L. Graetz, On the heat capacity of fluids, *Ann. d. Phys* **18**, 79–94 (1883).
2. L. Graetz, On the heat capacity of fluids, *Ann. d. Phys* **25**, 337–357 (1885).
3. W. Nusselt, The dependence of the heat-transfer coefficient on tube length, *Zeit. V.D.I.* **54**, 1154–1158 (1910).
4. M. Abramowitz, On the solution of the differential equation occurring in the problem of heat convection in laminar flow through a tube, *J. Math. Phys.* **32**, 184–187 (1953).
5. S. N. Singh, Heat transfer by laminar flow in a cylindrical tube, *Appl. Sci. Res.* **A7**, 325–340 (1958).

Table 5. *Mean mixed temperatures and Nusselt numbers*

	$Pe = 50$		100				500		1000			
ξ	$1 - \theta_M$	Nu	$1 - \theta_M$	Nu	Singh		$1 - \theta_M$	Nu	$1 - \theta_M$	Nu	Singh	
0·010	0·892	7·650	0·893	7·508	0·91	8·079	0·893	7·468	0·893	7·466	0·90	8·077
0·025	0·812	5·657	0·812	5·624	0·82	5·976	0·812	5·613	0·812	5·613	0·82	5·975
0·05	0·716	4·657	0·716	4·644	0·73	4·759	0·716	4·640	0·716	4·640	0·73	4·758
0·1	0·579	4·011	0·579	4·006	0·59	4·034	0·579	4·005	0·579	4·005	0·59	4·033
0·15	0·477	3·795	0·477	3·793	0·48	3·803	0·477	3·792	0·477	3·792	0·48	3·803
0·2	0·395	3·712	0·395	3·710	0·40	3·715	0·395	3·710	0·395	3·710	0·40	3·715
0·25	0·329	3·679	0·329	3·678	0·33	3·680	0·329	3·678	0·329	3·678	0·33	3·680
0·5	0·132	3·657	0·132	3·657	0·134	3·66	0·132	3·657	0·132	3·657	0·134	3·66
1	0·021	3·657	0·021	3·657	0·02	3·66	0·021	3·657	0·021	3·657	0·02	3·66

EXTENSIONS APPORTÉES À LA SOLUTION DU PROBLÈME DE GRAETZ

Résumé—Dans le travail original de Graetz sur le transfert thermique dans un écoulement laminaire, on fait une hypothèse simplificatrice pour laquelle est négligée la diffusion axiale par comparaison aux autres termes de l'équation. Ceci donne des résultats qui sont suffisamment précis pour les grandes valeurs du nombre de Péclét correspondant aux fluides classiques, mais l'hypothèse ne peut être appliquée au transfert thermique dans les métaux liquides où la diffusion axiale joue un rôle sensible. Des mémoires précédents tenant compte de la diffusion impliquent des calculs séparés pour chaque valeur du nombre de Péclét, tandis que dans cet article les valeurs propres sont données sous forme d'un développement asymptotique si bien que les valeurs cherchées peuvent être calculées de façon simple. La solution donnée ici tient aussi compte du préchauffage du fluide entrant et montre que cela a un effet significatif.

ERWEITERUNGEN DER LÖSUNG DES GRAETZ PROBLEMS

Zusammenfassung—In der Originalarbeit von Graetz über den Wärmeübergang bei laminarer Strömung wurde die vereinfachende Annahme getroffen, dass die axiale Wärmeleitung im Vergleich zu den anderen Gliedern der Gleichung vernachlässigt werden könne. Man erhält so genügend genaue Ergebnisse für Fluide mit grossen Pecletzahlen wie das bei klassischen Flüssigkeiten zutrifft.

Diese Annahme kann jedoch nicht gemacht werden für den Wärmeübergang bei flüssigen Metallen, wo der axialen Wärmeleitung eine bedeutende Rolle zukommt. Frühere Veröffentlichungen, die die Wärmeleitung berücksichtigen, benützten getrennte Berechnungen für jeden Wert der Pecletzahl. Diese Arbeit dagegen stellt die Eigenwerte in der Form einer asymptotischen Erweiterung dar, sodass die benötigten Werte in einfacher Weise berechnet werden können. Die hier dargestellte Lösung berücksichtigt auch die Vorheizung des einströmenden Fluids und zeigt, dass sie einen deutlichen Einfluss hat.

ДАЛЬНЕЙШЕЕ РАЗВИТИЕ РЕШЕНИЯ ЗАДАЧИ ГРЕТЦА

Аннотация—В оригинальной работе Гретца по теплообмену при ламинарном течении сделано упрощающее допущение, что в уравнении можно пренебречь осевым диффузионным членом по сравнению с другими диффузионными членами уравнения. Это позволило получить результаты, которые достаточно точны при больших числах Пекле для классических жидкостей. Однако, это допущение не может быть применено к жидким металлам, где аксиальная диффузия играет важную роль. В предыдущих работах, учитывающих диффузию, проводился отдельно расчет по каждому значению числа Пекле, в то время как в этой работе собственные значения даются в виде асимптотического разложения, что упрощает расчет искомых величин. В решении, предлагаемом в данной работе, учитывается также подогрев входящего потока и показывается его значительный эффект.